

ONE-DIMENSIONAL PROJECTIVE STRUCTURES, CONVEX CURVES AND THE OVALS OF BENGURIA & LOSS

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ABSTRACT. Benguria and Loss have conjectured that, amongst all smooth closed curves in \mathbb{R}^2 of length 2π , the lowest possible eigenvalue of the operator $L = -\Delta + \kappa^2$ was 1. They observed that this value was achieved on a two-parameter family, \mathcal{O} , of geometrically distinct ovals containing the round circle and collapsing to a multiplicity-two line segment. We characterize the curves in \mathcal{O} as absolute minima of two related geometric functionals. We also discuss a connection with projective differential geometry and use it to explain the natural symmetries of all three problems.

1. INTRODUCTION

In [1], Benguria and Loss conjectured that for any, σ , a smooth closed curve in \mathbb{R}^2 of length 2π , the lowest eigenvalue, λ_σ , of the operator $L_\sigma = -\Delta_\sigma + \kappa_\sigma^2$ satisfied $\lambda_\sigma \geq 1$. That is, they conjectured that for all such σ and all functions $f \in H^1(\sigma)$,

$$(1.1) \quad \int_\sigma |\nabla_\sigma f|^2 + \kappa_\sigma^2 f^2 \, ds \geq \int_\sigma f^2 \, ds,$$

where $\nabla_\sigma f$ is the intrinsic gradient of f , κ_σ is the geodesic curvature and ds is the length element. This conjecture was motivated by their observation that it was equivalent to a certain one-dimensional Lieb-Thirring inequality with conjectured sharp constant. They further observed that the above inequality is saturated on a two-parameter family of strictly convex curves which contains the round circle and degenerates into a multiplicity-two line segment. The curves in this family look like ovals and so we call them the ovals of Benguria and Loss and denote the family by \mathcal{O} . Finally, they showed that for closed curves $\lambda_\sigma \geq \frac{1}{2}$.

Further work on the conjecture was carried out by Burchard and Thomas in [3]. They showed that λ_σ is strictly minimized in a certain neighborhood of \mathcal{O} in the space of closed curves – verifying the conjecture in this neighborhood. More globally, Linde [6] improved the lower bound to $\lambda_\sigma \geq 0.608$ when σ is a planar convex curves. In addition, he showed that $\lambda_\sigma \geq 1$ when σ satisfied a certain symmetry condition. Recently, Denzler [4] has shown that if the conjecture is false, then the infimum of λ_σ over the space of closed curves is achieved by a closed strictly convex planar curve. Coupled with Linde's work, this implies that for any closed curve $\lambda_\sigma \geq 0.608$. In a different direction, the first author and Breiner in [2] connected the ovals conjecture to a certain convexity property for the length of curves in a minimal annulus.

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In the present article, we consider the family \mathcal{O} and observe that the curves in this class are the unique minimizers of two natural geometric functionals. To motivate these functionals, we first introduce a spectral functional modeled on (1.1). Specifcally, for a smooth curve, σ , of length $L(\sigma)$ and function, $f \in C^\infty(\sigma)$, set

$$(1.2) \quad \mathcal{E}_S[\sigma, f] = \int_\sigma |\nabla_\sigma f|^2 + \kappa_\sigma^2 f^2 - \frac{(2\pi)^2}{L(\sigma)^2} f^2 \, ds.$$

Clearly, the conjecture of Benguria and Loss is equivalent to the non-negativity of this functional. For any strictly convex smooth curve, σ , we introduce two related purely geometric functionals

$$(1.3) \quad \mathcal{E}_G[\sigma] = \int_\sigma \frac{|\nabla_\sigma \kappa_\sigma|^2}{4\kappa_\sigma^3} - \frac{(2\pi)^2}{L(\sigma)^2} \frac{1}{\kappa_\sigma} \, ds + 2\pi,$$

and

$$(1.4) \quad \mathcal{E}_G^*[\sigma] = \int_\sigma \frac{|\nabla_\sigma \kappa_\sigma|^2}{4\kappa_\sigma^2} - \kappa_\sigma^2 \, ds + \frac{(2\pi)^2}{L(\sigma)}.$$

As we will show, the geometric functionals \mathcal{E}_G and \mathcal{E}_G^* are dual to each other in a certain sense – which justifies the notation.

Our main result is that the functionals (1.3) and (1.4) are always non-negative and are zero only for ovals.

Theorem 1.1. *If σ is a smooth strictly convex closed curve in \mathbb{R}^2 , then both $\mathcal{E}_G[\sigma] \geq 0$ and $\mathcal{E}_G^*[\sigma] \geq 0$ with equality if and only if $\sigma \in \mathcal{O}$.*

To the best of our knowledge both inequalities are new. Clearly,

$$\mathcal{E}_G[\sigma] = \mathcal{E}_S[\sigma, \kappa_\sigma^{-1/2}],$$

and so the non-negativity of (1.3) would follow from the non-negativity of (1.2). Hence, Theorem 1.1 provides evidence for the conjecture of Benguria and Loss.

We also discuss the natural symmetry of these functionals. In particular, we show there are natural actions of $\mathrm{SL}(2, \mathbb{R})$ on \mathcal{D}^∞ , the space of smooth, degree-one curves and on \mathcal{D}_+^∞ , the space of smooth strictly convex degree-one curves, which preserve the functionals. A (possibly open) smooth planar curve is *degree-one* if its unit tangent map is a degree one map from \mathbb{S}^1 to \mathbb{S}^1 – for instance, any closed convex curve. A degree-one curve is *strictly convex* if the unit tangent map is injective.

Theorem 1.2. *There are actions of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{D}^\infty \times C^\infty$, the domain of \mathcal{E}_S , and on \mathcal{D}_+^∞ the domain of \mathcal{E}_G and \mathcal{E}_G^* so that for $L \in \mathrm{SL}(2, \mathbb{R})$*

$$\mathcal{E}_S[(\sigma, f) \cdot L] = \mathcal{E}_S[\sigma, f], \quad \mathcal{E}_G[\sigma \cdot L] = \mathcal{E}_G[\sigma], \quad \text{and} \quad \mathcal{E}_G^*[L \cdot \sigma] = \mathcal{E}_G^*[\sigma].$$

Furthermore, there is an involution $$: $\mathcal{D}_+^\infty \rightarrow \mathcal{D}_+^\infty$ so that*

$$*(L \cdot \sigma) = * \sigma \cdot L^{-1} \quad \text{and} \quad \mathcal{E}_G[*\sigma] = \mathcal{E}_G^*[\sigma].$$

We observe that \mathcal{O} is precisely the orbit of the round circle under these actions. Generically, the action does not preserve the condition of being a closed curve. Indeed, the image of the set of closed curves under this action is an open set in the space of curves and so is not well suited for the direct method in the calculus of variations. Arguably, this is the source of the difficulty in answering the conjecture. Indeed, we prove Theorem 1.1 in part by overcoming it.

2. PRELIMINARIES

Denote by $\mathbb{S}^1 = \{x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$ the unit circle in \mathbb{R}^2 . Unless otherwise stated, we always assume that \mathbb{S}^1 inherits the standard orientation from \mathbb{R}^2 and consider $d\theta$ to be the associated volume form and ∂_θ the dual vector field. Abusing notation slightly, let $\theta : \mathbb{S}^1 \rightarrow [0, 2\pi)$ be the compatible chart with $\theta(\mathbf{e}_1) = 0$. Let $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ be the covering map so that $\pi^*d\theta = dx$ and $\pi(0) = \mathbf{e}_1$ – here x is the usual coordinate on \mathbb{R} . Denote by $I : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ the involution given by $I(p) = -p$. Hence, $\theta(I(p)) = \theta(p) + \pi \pmod{2\pi}$.

Definition 2.1. An immersion $\sigma : [0, 2\pi] \rightarrow \mathbb{R}^2$ is a *degree-one curve of class $C^{k+1, \alpha}$* , if there is

- a degree-one map $\mathbf{T}_\sigma : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of class $C^{k, \alpha}$, the *unit tangent map of σ* ,
- a point $\mathbf{x}_\sigma \in \mathbb{R}^2$, the *base point of σ* , and
- a value $L(\sigma) > 0$, the *length of σ* ,

so that

$$\sigma(t) = \mathbf{x}_\sigma + \frac{L(\sigma)}{2\pi} \int_0^t \mathbf{T}_\sigma(\pi(x)) \, dx.$$

The curve σ is *strictly convex* provided the unit tangent map \mathbf{T}_σ is injective and is *closed* provided $\sigma(0) = \sigma(2\pi)$.

A degree-one curve, σ , is uniquely determined by the data $(\mathbf{T}_\sigma, \mathbf{x}_\sigma, L(\sigma))$. Denote by $\mathcal{D}^{k+1, \alpha}$ the set of degree-one curves of class $C^{k+1, \alpha}$ and by $\mathcal{D}_+^{k+1, \alpha} \subset \mathcal{D}^{k+1, \alpha}$ the set of strictly convex degree-one curves of class $C^{k+1, \alpha}$. The length element associated to σ is $ds = \frac{L(\sigma)}{2\pi} dx = \frac{L(\sigma)}{2\pi} \pi^*d\theta = \pi^*\widetilde{ds}$. If $\sigma \in \mathcal{D}^2$, then the geodesic curvature, κ_σ , of σ satisfies $\kappa_\sigma = \pi^*\tilde{\kappa}_\sigma$ where $\tilde{\kappa}_\sigma \in C^{k-1, \alpha}(\mathbb{S}^1)$ satisfies

$$\int_{\mathbb{S}^1} \tilde{\kappa}_\sigma \, \widetilde{ds} = 2\pi.$$

Conversely, given such a κ_σ there is a degree-one curve with geodesic curvature κ_σ . Abusing notation slightly, we will not distinguish between ds and \widetilde{ds} and between κ_σ and $\tilde{\kappa}_\sigma$. Clearly, $\sigma \in \mathcal{D}_+^2$ if and only if $\kappa_\sigma > 0$.

The *standard parameterization of \mathbb{S}^1* will be the parameterization of \mathbb{S}^1 with data $(\mathbf{T}_0, \mathbf{e}_1, 2\pi)$ where

$$\mathbf{T}_0(p) = -\sin(\theta(p))\mathbf{e}_1 + \cos(\theta(p))\mathbf{e}_2.$$

Let $\text{Diff}_+^{k, \alpha}(\mathbb{S}^1)$ denote the orientation preserving diffeomorphisms of \mathbb{S}^1 of class $C^{k, \alpha}$ – that is bijective maps of class $C^{k, \alpha}$ with inverse of class $C^{k, \alpha}$. Endow this space with the usual $C^{k, \alpha}$ topology. If $\sigma \in \mathcal{D}_+^{k+1, \alpha}$, then there is a $\phi_\sigma \in \text{Diff}_+^{k, \alpha}(\mathbb{S}^1)$, the *induced diffeomorphism of σ* so that the unit tangent map of σ is

$$\mathbf{T}_\sigma(p) = \mathbf{T}_0(\phi_\sigma(p)).$$

The induced diffeomorphism of the standard parameterization of \mathbb{S}^1 is the identity map. For $\phi \in \text{Diff}_+^1(\mathbb{S}^1)$, let $\phi' \in C^0(\mathbb{S}^1)$ be the function so $\phi^*d\theta = \phi'd\theta$. Observe that $\phi' > 0$. Similarly, for any function $f \in C^1(\mathbb{S}^1)$ set $f' = \partial_\theta f \in C^0(\mathbb{S}^1)$. If $\phi \in \text{Diff}_+^k(\mathbb{S}^1)$, then inductively define $\phi^{(k)} = (\phi^{(k-1)})'$. A simple computation shows that if $\sigma \in \mathcal{D}_+^2$, then $\phi'_\sigma = \kappa_\sigma \frac{L(\sigma)}{2\pi}$.

Consider the group homomorphism

$$\Gamma : \text{SL}(2, \mathbb{R}) \rightarrow \text{Diff}_+^\infty(\mathbb{S}^1)$$

given by

$$\Gamma(L) = \mathbf{x} \mapsto \frac{L \cdot \mathbf{x}}{|L \cdot \mathbf{x}|}.$$

where $\mathbf{x} \in \mathbb{S}^1$ and $L \in \mathrm{SL}(2, \mathbb{R})$. Denote the image of Γ by $\mathrm{Möb}(\mathbb{S}^1)$ which we refer to as the Möbius group of \mathbb{S}^1 . One computes that

$$\mathbf{T}_0 \circ \Gamma(L) = \frac{L \cdot \mathbf{T}_0}{|L \cdot \mathbf{T}_0|}.$$

Which are precisely the unit tangent maps of the ovals that Benguria and Loss introduced in [1]. That is,

$$\mathcal{O} = \{\sigma \in \mathcal{D}_+^\infty : \phi_\sigma \in \mathrm{Möb}(\mathbb{S}^1)\}.$$

3. PROJECTIVE SYMMETRIES

We review some basic concepts from projective differential geometry which will motivate the definition of $\mathrm{Möb}(\mathbb{S}^1)$ made above as well as provide the natural context for the symmetries of the functionals of (1.2), (1.3) and (1.4). This is a vast subject with many different perspectives and we present only a summarized version we refer the interested reader to the excellent book [7] by Ovsienko and Tabachnikov as well as their article [8] – these were our main sources for this material.

3.1. One-Dimensional Projective Differential Geometry. Let M be a one-dimensional oriented manifold. We denote by $\Omega^s(M)$ the space of weight s densities on M , so $\Omega^2(M)$ can be identified with the space of (symmetric) $(0, 2)$ -tensors and, using the orientation, $\Omega^1(M)$ can be identified with the space of one-forms. There is a canonical tensor product

$$\otimes : \Omega^s(M) \times \Omega^t(M) \rightarrow \Omega^{s+t}(M).$$

For any affine connection ∇ on M , the orientation may be used to construct associated first order differential operators

$$\nabla : \Omega^s(M) \rightarrow \Omega^{s+1}(M).$$

A *real projective structure*, \mathcal{P} on M is a second-order elliptic differential operator

$$\mathcal{P} : \Omega^{-1/2}(M) \rightarrow \Omega^{3/2}(M)$$

so that there is some affine connection ∇ on M and $P \in \Omega^2(M)$ with

$$\mathcal{P} = \nabla^2 + P.$$

One verifies that, given two real projective structures \mathcal{P}_1 and \mathcal{P}_2 , $\mathcal{P}_2 - \mathcal{P}_1 \in \Omega^2(M)$ is a zero-order operator. Hence, the space of real projective structures is an affine space modeled on $\Omega^2(M)$. Given an orientation preserving smooth diffeomorphism $\phi : M_1 \rightarrow M_2$ we define the push forward and pull backs of real projective structures \mathcal{P}_i on M_i in an obvious fashion. That is,

$$(\phi_* \mathcal{P}_1) \cdot \theta = (\phi^{-1})^* (\mathcal{P}_1 \cdot \phi^* \theta) \quad \text{and} \quad (\phi^* \mathcal{P}_2) \cdot \theta = \phi^* (\mathcal{P}_2 \cdot (\phi^{-1})^* \theta).$$

The *Schwarzian derivative* of ϕ relative to $\mathcal{P}_1, \mathcal{P}_2$ is

$$S_{\mathcal{P}_1, \mathcal{P}_2}(\phi) = \phi^* \mathcal{P}_2 - \mathcal{P}_1 \in \Omega^2(M_1).$$

The Schwarzian satisfies the following co-cycle condition

$$S_{\mathcal{P}_1, \mathcal{P}_3}(\phi_2 \circ \phi_1) = \phi_1^* S_{\mathcal{P}_2, \mathcal{P}_3}(\phi_2) + S_{\mathcal{P}_1, \mathcal{P}_2}(\phi_1).$$

Given a $\phi \in \text{Diff}_+^\infty(M)$ and a real projective structure \mathcal{P} write $S_{\mathcal{P}}(\phi) = S_{\mathcal{P}, \mathcal{P}}(\phi)$. An orientation preserving diffeomorphism ϕ is a *Möbius transformation of \mathcal{P}* if and only if $S_{\mathcal{P}}(\phi) = 0$. The co-cycle condition implies that the set of such maps forms a subgroup, $\text{Möb}(\mathcal{P})$, of $\text{Diff}_+^\infty(M)$.

Let \mathbb{RP}^1 be the one-dimensional real projective space – in other words the space of unoriented lines through the origin in \mathbb{R}^2 . Let (x_1, x_2) be the usual linear coordinates on \mathbb{R}^2 . If $(x_1, x_2) \neq 0$, then we denote by $[x_1 : x_2]$ the point in \mathbb{RP}^1 corresponding to the line through the origin and (x_1, x_2) . On the chart $U = \{[x_1 : x_2] : x_2 \neq 0\}$ we have the affine coordinate $\tau = x_1/x_2$ for \mathbb{RP}^1 . Let ${}^\tau\nabla$ be the (unique) connection so that ∂_τ is parallel. There is a unique real projective structure $\mathcal{P}_{\mathbb{RP}^1}$ on \mathbb{RP}^1 so that $\mathcal{P}_{\mathbb{RP}^1} = {}^\tau\nabla^2$. This is the *standard real projective structure* on \mathbb{RP}^1 .

If $\phi \in \text{Diff}_+^\infty(\mathbb{RP}^1)$, then one computes that

$$S_{\mathbb{RP}^1}(\phi) = S_{\mathcal{P}_{\mathbb{RP}^1}}(\phi) = \left(\frac{\phi'''}{\phi'} - \frac{3}{2} \left(\frac{\phi''}{\phi'} \right)^2 \right) d\tau^2$$

where here

$$\phi' = \partial_\tau(\tau \circ \phi)$$

and likewise for the higher derivatives. This is the classical form of the Schwarzian derivative. Write $\text{Möb}(\mathbb{RP}^1)$ for the Möbius group of $\mathcal{P}_{\mathbb{RP}^1}$ and observe these are the fractional linear transformations. Indeed, if $\phi \in \text{Möb}(\mathbb{RP}^1)$, then there is a matrix

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$$

so that

$$\tau(\phi(p)) = \frac{a\tau(p) + b}{c\tau(p) + d}.$$

This corresponds to the natural action of $\text{SL}(2, \mathbb{R})$ on the space of lines through the origin. Let

$$\gamma : \text{SL}(2, \mathbb{R}) \rightarrow \text{Diff}_+^\infty(\mathbb{RP}^1).$$

denote this group homomorphism. Notice that $\ker \rho = \pm \text{Id}$ and so this map induces an injective homomorphism

$$\tilde{\gamma} : \text{PSL}(2, \mathbb{R}) \rightarrow \text{Diff}_+^\infty(\mathbb{RP}^1)$$

whose image is $\text{Möb}(\mathbb{RP}^1)$.

Consider the natural map $T : \mathbb{S}^1 \rightarrow \mathbb{RP}^1$ given by sending a point p to the tangent line to \mathbb{S}^1 through p . Let ${}^\theta\nabla$ be the unique connection on \mathbb{S}^1 so that ∂_θ is parallel and let $\mathcal{D}_{\mathbb{S}^1} = {}^\theta\nabla^2$. If $\phi \in \text{Diff}_+^\infty(\mathbb{S}^1)$, then one computes that

$$S_{\mathbb{S}^1}(\phi) = S_{\mathcal{D}_{\mathbb{S}^1}}(\phi) = \left(\frac{\phi'''}{\phi'} - \frac{3}{2} \left(\frac{\phi''}{\phi'} \right)^2 \right) d\theta^2$$

where here ϕ' has already been defined. Define $S_\theta(\phi)$ so $S_{\mathbb{S}^1}(\phi) = S_\theta(\phi)d\theta^2$.

As $T \circ I = T$, if $\phi \in \text{Diff}_+^\infty(\mathbb{S}^1)$ satisfies $\phi \circ I = I \circ \phi$, then there is a well-defined element $\tilde{T}(\phi) \in \text{Diff}_+^\infty(\mathbb{RP}^1)$ so that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\phi} & \mathbb{S}^1 \\ \downarrow T & & \downarrow T \\ \mathbb{RP}^1 & \xrightarrow{\tilde{T}(\phi)} & \mathbb{RP}^1 \end{array}$$

A straightforward computation shows that,

$$S_{\mathbb{S}^1, \mathbb{RP}^1}(T) = S_{\mathcal{P}_{\mathbb{S}^1}, \mathcal{P}_{\mathbb{RP}^1}}(T) = 2d\theta^2.$$

Hence, for a $\phi \in \text{Diff}_+^\infty(\mathbb{S}^1)$ which satisfies $\phi \circ I = I \circ \phi$ the co-cycle relation for the Schwarzian implies

$$\begin{aligned} 0 &= S_{\mathbb{S}^1, \mathbb{RP}^1}(T \circ \phi) - S_{\mathbb{S}^1, \mathbb{RP}^1}(\tilde{T}(\phi) \circ T) \\ &= 2\phi^* d\theta^2 + S_{\mathbb{S}^1}(\phi) - T^* S_{\mathbb{RP}^1}(\tilde{T}(\phi)) + 2d\theta^2 \\ &= S_{\mathbb{S}^1}(\phi) + 2(\phi')^2 d\theta^2 + 2d\theta^2 - T^* S_{\mathbb{RP}^1}(\tilde{T}(\phi)) \end{aligned}$$

That is,

$$(3.1) \quad S_{\mathbb{S}^1}(\phi) + 2(\phi')^2 d\theta^2 - 2d\theta^2 = T^* S_{\mathbb{RP}^1}(\tilde{T}(\phi)).$$

One verifies from their definitions that $\tilde{T}(\text{Möb}(\mathbb{S}^1)) = \text{Möb}(\mathbb{RP}^1)$. Hence, (3.1) gives a differential characterization of $\text{Möb}(\mathbb{S}^1)$. Finally, we note the following commutative diagram

$$\begin{array}{ccc} \text{SL}(2, \mathbb{R}) & \xrightarrow{\Gamma} & \text{Möb}(\mathbb{S}^1) \\ \downarrow \pi & \searrow \gamma & \downarrow \tilde{T} \\ \text{PSL}(2, \mathbb{R}) & \xrightarrow{\tilde{\gamma}} & \text{Möb}(\mathbb{RP}^1) \end{array}$$

where π is the natural projection.

Remark 3.1. We have defined a real projective structure on M in terms of a differential operator. Equivalently (and more commonly), a real projective structure on M may be defined to be a maximal atlas mapping open sets in M into \mathbb{RP}^1 such that the transition functions are restriction of fractional linear transformations. For the equivalency of the two definitions the reader may consult [7].

3.2. Projective Symmetries. We now describe the natural symmetries of (1.2), (1.3) and (1.4). We also introduce a notion of duality for strictly convex degree-one curves – this duality will streamline some of the arguments and may be of independent interest. To begin for $\sigma \in \mathcal{D}^{k+1, \alpha}$ we define the *dual curve*, $\sigma^* \in \mathcal{D}^{k+1, \alpha}$ to be the unique curve with

$$\phi_{\sigma^*} = \phi_\sigma^{-1}, \quad \mathbf{x}_{\sigma^*} = \mathbf{x}_\sigma \quad \text{and} \quad L(\sigma^*) = L(\sigma).$$

That is, σ^* is the curve whose induced diffeomorphism is the inverse to the induced diffeomorphism of σ . Clearly, $(\sigma^*)^* = \sigma$. To proceed further, we note that the functionals (1.3) and (1.4) can, by an integration by parts, be made to naturally

involve the Schwarzian derivative. To that end pick a $\sigma \in \mathcal{D}_+^4$. By scaling, we may assume that $L(\sigma) = 2\pi$. Hence, $\kappa_\sigma = \phi'_\sigma$ and so

$$\begin{aligned}
 \mathcal{E}_G[\sigma] &= \int_{\mathbb{S}^1} \frac{(\phi''_\sigma)^2}{4(\phi'_\sigma)^3} - \frac{1}{\phi'_\sigma} + \phi'_\sigma \, d\theta \\
 (3.2) \quad &= \int_{\mathbb{S}^1} \left(\frac{\phi''_\sigma}{2(\phi'_\sigma)^3} \right)' \phi'_\sigma + \frac{3(\phi''_\sigma)^2}{4(\phi'_\sigma)^3} - \frac{1}{\phi'_\sigma} + \phi'_\sigma \, d\theta \\
 &= \frac{1}{2} \int_{\mathbb{S}^1} \frac{S_\theta(\phi_\sigma) + 2(\phi'_\sigma)^2 - 2}{\phi'_\sigma} \, d\theta
 \end{aligned}$$

where the second equality follows by integrating by parts. Likewise, a straightforward integration by parts gives

$$(3.3) \quad \mathcal{E}_G^*[\sigma] = -\frac{1}{2} \int_{\mathbb{S}^1} S_\theta(\phi_\sigma) + 2(\phi'_\sigma)^2 - 2 \, d\theta.$$

An immediate consequence of this is the following useful fact,

Proposition 3.2. *For $\sigma \in \mathcal{D}_+^4$, $\mathcal{E}_G[\sigma] = \mathcal{E}_G^*[\sigma^*]$.*

Proof. By scaling we may assume that $L(\sigma) = 2\pi$. Write $\psi_\sigma = \phi_\sigma^{-1}$. The co-cycle property for the Schwarzian implies that

$$S_\theta(\psi_\sigma) = -\frac{S_\theta(\phi_\sigma) \circ \phi_\sigma^{-1}}{(\phi'_\sigma \circ \phi_\sigma^{-1})^2}.$$

Where we have used that

$$\phi'_\sigma = \frac{1}{\psi'_\sigma \circ \phi_\sigma}.$$

Hence, by (3.2) and (3.3)

$$\begin{aligned}
 \mathcal{E}_G[\sigma] &= \frac{1}{2} \int_{\mathbb{S}^1} \frac{S_\theta(\phi_\sigma) + 2(\phi'_\sigma)^2 - 2}{\phi'_\sigma} \, d\theta \\
 &= -\frac{1}{2} \int_{\mathbb{S}^1} (S_\theta(\psi_\sigma) \circ \phi_\sigma) \phi'_\sigma + 2(\psi'_\sigma \circ \phi_\sigma)^2 \phi'_\sigma - 2\phi'_\sigma \, d\theta \\
 &= -\frac{1}{2} \int_{\mathbb{S}^1} S_\theta(\psi_\sigma) + 2(\psi'_\sigma)^2 - 2 \, d\theta \\
 &= \mathcal{E}_G^*[\sigma^*].
 \end{aligned}$$

□

We now may define the desired actions. Consider first the right action of $\text{Möb}(\mathbb{S}^1)$ on $\mathcal{D}^{k+1,\alpha}$, $\sigma \cdot \varphi = \sigma'$ where $\sigma' \in \mathcal{D}^{k+1,\alpha}$ is the unique element with

$$\mathbf{T}_{\sigma'} = \mathbf{T}_\sigma \circ \varphi, \quad \mathbf{x}_{\sigma'} = \mathbf{x}_\sigma \quad \text{and} \quad L(\sigma') = L(\sigma).$$

Notice, that if $\sigma \in \mathcal{D}_+^{k+1,\alpha}$ is strictly convex, then so is σ' and in this case we have that $\phi_{\sigma'} = \phi_\sigma \circ \varphi$. With this in mind, we also consider a left action of $\text{Möb}(\mathbb{S}^1)$ on $\mathcal{D}_+^{k+1,\alpha}$, $\varphi \cdot \sigma = \sigma'$, where $\sigma' \in \mathcal{D}_+^{k+1,\alpha}$ is the unique element with

$$\phi_{\sigma'} = \varphi \circ \phi_\sigma, \quad \mathbf{x}_{\sigma'} = \mathbf{x}_\sigma \quad \text{and} \quad L(\sigma') = L(\sigma).$$

We observe that for $\sigma \in \mathcal{D}_+^{k+1}$ and $\varphi \in \text{Möb}(\mathbb{S}^1)$,

$$\varphi \cdot \sigma^* = (\sigma \cdot \varphi^{-1})^*.$$

Finally, we define a right action of $\text{Möb}(\mathbb{S}^1)$ on $C^{k,\alpha}(\mathbb{S}^1)$ by

$$f \cdot \varphi = (\varphi')^{-1/2} f \circ \varphi.$$

If we use $d\theta$ to identify $C^\infty(\mathbb{S}^1)$ with $\Omega^{-1/2}(\mathbb{S}^1)$, then this is the natural pull-back action on $\Omega^{-1/2}(\mathbb{S}^1)$.

Proposition 3.3. *For any $\varphi \in \text{Möb}(\mathbb{S}^1)$, $\sigma \in \mathcal{D}^\infty$ and $f \in C^\infty(\mathbb{S}^1)$,*

$$\mathcal{E}_S[\sigma, f] = \mathcal{E}_S[\sigma \cdot \varphi, f \cdot \varphi].$$

Likewise, for any $\varphi \in \text{Möb}(\mathbb{S}^1)$ and $\sigma \in \mathcal{D}_+^\infty$,

$$\mathcal{E}_G[\sigma] = \mathcal{E}_G[\sigma \cdot \varphi] \quad \text{and} \quad \mathcal{E}_G^*[\sigma] = \mathcal{E}_G^*[\varphi \cdot \sigma].$$

Proof. By scaling, it suffices to take $L(\sigma) = 2\pi$ so $ds = d\theta$. Set

$$f'_\varphi = (\varphi')^{1/2} f' \circ \varphi - \frac{1}{2} \frac{\varphi''}{(\varphi')^{3/2}} f \circ \varphi$$

We show the first symmetry by computing,

$$\begin{aligned} (f'_\varphi)^2 &= (f' \circ \varphi)^2 \varphi' - \frac{\varphi''}{\varphi'} (f' \circ \varphi)(f \circ \varphi) + \frac{1}{4} \frac{(\varphi'')^2}{(\varphi')^3} (f \circ \varphi)^2 \\ &= (f' \circ \varphi)^2 \varphi' - \frac{1}{2} \frac{\varphi''}{(\varphi')^2} \partial_\theta (f \circ \varphi)^2 + \frac{1}{4} \frac{(\varphi'')^2}{(\varphi')^2} f_\varphi^2 \\ &= (f' \circ \varphi)^2 \varphi' - \partial_\theta \left(\frac{\varphi'' (f \circ \varphi)^2}{2(\varphi')^2} \right) + \left(\frac{\varphi''}{2(\varphi')^2} \right)' (f \circ \varphi)^2 + \frac{1}{4} \frac{(\varphi'')^2}{(\varphi')^2} f_\varphi^2 \\ &= (f' \circ \varphi)^2 \varphi' - \partial_\theta \left(\frac{\varphi'' (f \circ \varphi)^2}{2(\varphi')^2} \right) + \frac{1}{2} S_\theta(\varphi) f_\varphi^2 \\ &= (f' \circ \varphi)^2 \varphi' - \partial_\theta \left(\frac{\varphi'' (f \circ \varphi)^2}{2(\varphi')^2} \right) + (1 - (\varphi')^2) f_\varphi^2 + \frac{1}{2} (S_\theta(\varphi) + 2(\varphi')^2 - 2) f_\varphi^2 \\ &= (f' \circ \varphi)^2 \varphi' - \partial_\theta \left(\frac{\varphi'' (f \circ \varphi)^2}{2(\varphi')^2} \right) + (1 - (\varphi')^2) f_\varphi^2. \end{aligned}$$

The last inequality used $\varphi \in \text{Möb}(\mathbb{S}^1)$ and (3.1). Integrating by parts gives,

$$\int_{\mathbb{S}^1} (f'_\varphi)^2 - f_\varphi^2 d\theta = \int_{\mathbb{S}^1} (f' \circ \varphi)^2 \varphi' - (f \circ \varphi)^2 \varphi' d\theta.$$

Hence, after a change of variables

$$\int_{\mathbb{S}^1} (f'_\varphi)^2 - f_\varphi^2 d\theta = \int_{\mathbb{S}^1} (f')^2 - f^2 d\theta$$

Finally,

$$(\kappa_\varphi f_\varphi)^2 = \kappa_\varphi(\varphi(\theta))^2 f \circ \varphi^2 \varphi'$$

and so a change of variables gives,

$$\int_{\mathbb{S}^1} (\kappa_\varphi f_\varphi)^2 d\theta = \int_{\mathbb{S}^1} \kappa^2 f^2 d\theta.$$

That is, $\mathcal{E}_S[\sigma, f] = \mathcal{E}_S[\sigma \cdot \varphi, f \cdot \varphi]$.

The co-cycle property of the Schwarzian and (3.3) immediately implies

$$\begin{aligned}\mathcal{E}_G^*[\varphi \cdot \sigma] &= -\frac{1}{2} \int_{\mathbb{S}^1} S_\theta(\varphi \circ \phi_\sigma) + 2((\varphi \circ \phi_\sigma)')^2 - 2 \, d\theta \\ &= -\frac{1}{2} \int_{\mathbb{S}^1} S_\theta(\phi_\sigma) - 2(\phi'_\sigma)^2(\varphi' \circ \phi_\sigma)^2 + 2(\phi'_\sigma)^2 + 2(\phi'_\sigma)^2(\varphi' \circ \phi_\sigma)^2 - 2 \, d\theta \\ &= \mathcal{E}_G^*[\sigma]\end{aligned}$$

Finally, using Proposition 3.2

$$\mathcal{E}_G[\sigma \cdot \varphi] = \mathcal{E}_G^*[(\sigma \cdot \varphi)^*] = \mathcal{E}_G^*[\varphi^{-1} \cdot \sigma^*] = \mathcal{E}_G^*[\sigma^*] = \mathcal{E}_G[\sigma].$$

□

Theorem 1.2 is an immediate consequence of Propositions 3.2 and 3.3 and the fact that $\text{Möb}(\mathbb{S}^1)$ is isomorphic to $\text{SL}(2, \mathbb{R})$.

As a final remark, we observe that we may extend the duality operator to $\mathcal{D}_+^\infty \times C^\infty(\mathbb{S}^1)$ and define a natural dual functional to \mathcal{E}_S . Namely, set

$$(\sigma, f)^* = (\sigma^*, f \circ \phi_\sigma^{-1})$$

and define

$$\mathcal{E}_S^*[\sigma, f] = \int_\sigma \frac{|\nabla_\sigma f|^2}{\kappa_\sigma} - \kappa_\sigma f^2 + \frac{(2\pi)^2}{L(\sigma)^2} \frac{f^2}{\kappa_\sigma} \, ds.$$

We then have,

Proposition 3.4. *If $\sigma \in \mathcal{D}_+^\infty$ and $f \in C^\infty(\mathbb{S}^1)$, then*

$$\mathcal{E}_S[(\sigma, f)^*] = \mathcal{E}_S^*[\sigma, f].$$

Proof. By scaling, we may assume that $L(\sigma) = 2\pi$. Writing $\psi_\sigma = \phi_\sigma^{-1}$, we compute

$$\begin{aligned}\mathcal{E}_S[(\sigma, f)^*] &= \int_{\mathbb{S}^1} ((f \circ \psi_\sigma)')^2 + (\psi'_\sigma)^2 (f \circ \psi_\sigma)^2 - (f \circ \psi_\sigma)^2 \, d\theta \\ &= \int_{\mathbb{S}^1} (\psi'_\sigma)^2 (f' \circ \psi_\sigma)^2 + (\psi'_\sigma)^2 (f \circ \psi_\sigma)^2 - \frac{(f \circ \psi_\sigma)^2}{\psi'_\sigma} \psi'_\sigma \, d\theta \\ &= \int_{\mathbb{S}^1} (\psi'_\sigma \circ \psi_\sigma^{-1}) (f' \circ \psi_\sigma)^2 + (\psi'_\sigma \circ \psi_\sigma^{-1}) (f \circ \psi_\sigma)^2 - \frac{(f \circ \psi_\sigma)^2}{\psi'_\sigma \circ \psi_\sigma^{-1}} \, d\theta \\ &= \int_{\mathbb{S}^1} \frac{(f' \circ \psi_\sigma)^2}{\phi'_\sigma} + \frac{(f \circ \psi_\sigma)^2}{\phi'_\sigma} - \phi'_\sigma (f \circ \psi_\sigma)^2 \, d\theta \\ &= \mathcal{E}_S^*[\sigma, f].\end{aligned}$$

□

4. DERIVING THE GEOMETRIC ESTIMATES

To prove Theorem 1.1 we will use the direct method in the calculus of variations on an appropriate subclass of the class of degree-one convex curves. This subclass is necessarily larger than the class of closed curves. We first note that the conjecture of Benguria and Loss holds for symmetric curves.

Proposition 4.1. *For $\sigma \in \mathcal{D}^2$, if the induced diffeomorphism satisfies $\phi_\sigma \circ I = I \circ \phi_\sigma$, then $\mathcal{E}_S[\sigma, f] \geq 0$ with equality if and only if $\sigma \in \mathcal{O}$ and $f = \kappa_\sigma^{-1/2}$ is the lowest eigenfunction of L_σ .*

Proof. By scaling we may assume $L(\sigma) = 2\pi$. The symmetry implies that $\kappa_\sigma \circ I = \kappa_\sigma$ and $\mathbf{T}_\sigma \circ I = -\mathbf{T}_\sigma$. Hence, $\mathcal{E}_S[\sigma, f] = \mathcal{E}_S[\sigma, f \circ I]$ and so, the variational characterization of the lowest eigenvalue implies that the lowest eigenfunction f must satisfy $f \circ I = f$. As observed in [1],

$$\mathcal{E}_S[\sigma, f] = \int_{\mathbb{S}^1} |\mathbf{y}'|^2 - |\mathbf{y}|^2 \, d\theta$$

where $\mathbf{y} = f\mathbf{T}_\sigma$. Moreover, $\mathbf{y}(p) = (a \cos \theta(p) + b \sin \theta(p), c \cos \theta(p) + d \sin \theta(p))$ if and only if $\sigma \in \mathcal{O}$. As $\mathbf{y} \circ I = -\mathbf{y}$,

$$\int_{\mathbb{S}^1} \mathbf{y} \, d\theta = 0$$

and the proposition follows from the one-dimensional Poincaré inequality. \square

Corollary 4.2. *For $\sigma \in \mathcal{D}_+^3$, if the induced diffeomorphism satisfies $\phi_\sigma \circ I = I \circ \phi_\sigma$, then $\mathcal{E}_G[\sigma] \geq 0$ with equality if and only if $\sigma \in \mathcal{O}$.*

Proof. Take $f = \kappa_\sigma^{-1/2}$ in (1.2) and use Proposition 4.1. \square

Motivated by [6], we make the following definition which is a weak analog of the preceding symmetry condition.

Definition 4.3. A point $p \in \mathbb{S}^1$ is a *balance point* of $\phi \in \text{Diff}_+^0$ if $\phi(I(p)) = I(\phi(p))$. If $\phi \in \text{Diff}_+^1(\mathbb{S}^1)$, then a balance point p of ϕ is *stable* if and only if $\phi'(p) \neq \phi'(I(p))$ and so – abusing terminology – is *unstable* if $\phi'(p) = \phi'(I(p))$.

Clearly, if p is a balance point then so is $I(p)$. Further, it follows from the intermediate value theorem that every $\phi \in \text{Diff}_+^0(\mathbb{S}^1)$ has at least one (and hence two) balance points. We emphasize that stable balance points of $\phi \in \text{Diff}_+^1(\mathbb{S}^1)$ are preserved by C^1 perturbations of ϕ – that is, the number of stable balance points is lower semi-continuous with respect to the C^1 topology.

The key observation of Linde [6, Lemma 2.1] is that closed convex curves have induced diffeomorphisms with a non-trivial number of balance points – we include the proof of this for the sake of completeness.

Lemma 4.4. *If $\psi \in \text{Diff}_+^1(\mathbb{S}^1)$ satisfies*

$$\int_{\mathbb{S}^1} \psi' \cos \theta \, d\theta = \int_{\mathbb{S}^1} \psi' \sin \theta \, d\theta = 0,$$

then ψ has at least four balance points.

Remark 4.5. Linde actually shows there are at least six balance points.

Proof. As $\int_{\mathbb{S}^1} \psi' \, d\theta = 2\pi$ and ψ' is continuous, there is a point p_0 , so that if γ_\pm are the components of $\mathbb{S}^1 \setminus \{p_0, I(p_0)\}$, then $\int_{\gamma_\pm} \psi' \, d\theta = \pi$ and so p_0 and $I(p_0)$ are balance points of ψ . We write ψ' as a Fourier series. That is, we write $\psi' = 1 + \sum_{n=2}^\infty (na_n \cos n\theta - nb_n \sin n\theta)$. By abuse of notation and a possible rotation, this means that $\psi = \theta + \sum_{n=2}^\infty (a_n \sin n\theta + b_n \cos n\theta)$ and that $\psi(0) + \pi = \psi(\pi)$. Up to relabelling, we identify $(0, \pi)$ with γ_+ and $(\pi, 2\pi)$ with γ_- . Write $\psi = \theta + f(\theta) + g(\theta)$ where f are the odd terms in the expansion and g are the even terms. Hence, $f(\theta + \pi) = -f(\theta)$, $g(\theta + \pi) = g(\theta)$ and $f(0) = 0 = f(\pi)$. Clearly, any balance point of ψ in γ_+ corresponds to a zero of f in $(0, \pi)$. If f does not change sign on this interval, then either $f \equiv 0$ and ψ has an infinite number of balance points, or

$\int_0^\pi f(\theta) \sin \theta \, d\theta \neq 0$. However, as $f(\theta + \pi) \sin(\theta + \pi) = f(\theta) \sin \theta$, this would imply $\int_0^{2\pi} f(\theta) \sin \theta \, d\theta \neq 0$ which is impossible. Hence, f must change sign and so f has at least one zero in $(0, \pi)$ which verifies the claim. \square

Corollary 4.6. *If $\sigma \in \mathcal{D}_+^2$ is closed, then ϕ_σ has at least four balance points.*

Proof. We may scale so $L(\sigma) = 2\pi$. If σ is closed, then $\int_{\mathbb{S}^1} \mathbf{T}_\sigma \, d\theta = 0$. That is, $\int_{\mathbb{S}^1} \mathbf{T}_0 \circ \phi_\sigma \, d\theta = 0$. Changing variables, gives $\int_{\mathbb{S}^1} (\phi_\sigma^{-1})' \mathbf{T}_0 \, d\theta$. Hence, ϕ_σ^{-1} has at least four balance points which immediately implies ϕ_σ does as well. \square

As we will see, the appropriate spaces to on which the functionals (1.3) and (1.4) have good lower bounds seem to be spaces of curves whose induced diffeomorphisms have non-trivial number of balance points. Motivated by this, set

$$\text{BDiff}_+^{k,\alpha}(\mathbb{S}^1, N) = \left\{ \phi \in \text{Diff}_+^{k,\alpha}(\mathbb{S}^1) : \phi \text{ has at least } N \text{ balance points} \right\}.$$

Hence, $\text{BDiff}_+^{k,\alpha}(\mathbb{S}^1, 2) = \text{Diff}_+^{k,\alpha}(\mathbb{S}^1)$ and $\text{Möb}(\mathbb{S}^1) \subset \text{BDiff}_+^\infty(\mathbb{S}^1, N)$ for every N . Denote by $\overline{\text{BDiff}}_+^{k,\alpha}(\mathbb{S}^1, N)$ the closure of $\text{BDiff}_+^{k,\alpha}(\mathbb{S}^1, N)$ inside of $\text{Diff}_+^{k,\alpha}(\mathbb{S}^1)$ and by $\partial \text{BDiff}_+^{k,\alpha}(\mathbb{S}^1, N)$ the topological boundary of $\text{BDiff}_+^{k,\alpha}(\mathbb{S}^1, N)$. We remark that there is a family of diffeomorphisms $\phi_\lambda \in \text{BDiff}_+^1(\mathbb{S}^1, 2)$ satisfying for $\lambda > 0$

$$\theta(\phi_\lambda(p)) = 2 \cot^{-1} \left(\lambda \cot \left(\frac{1}{2} \theta(p) \right) \right).$$

As $\lambda \rightarrow 1$ these converge to the identity map in the C^∞ topology and hence, the identity map is in $\partial \text{BDiff}_+^\infty(\mathbb{S}^1, 4)$. More generally, we observe

Lemma 4.7. *If $\phi \in \partial \text{BDiff}_+^1(\mathbb{S}^1, 4)$, then ϕ has at least one pair of unstable balance points.*

Proof. We note first that if ϕ has only two balance points, and these balance points were stable then by the lower semi-continuity, any C^1 perturbation of ϕ would still have only two balance points – that is $\phi \notin \overline{\text{BDiff}}_+^1(\mathbb{S}^1, 4)$. Hence, if ϕ has only two balance points, then they are unstable. If ϕ has at least four balance points, then at least one pair must be unstable as otherwise the lower semi-continuity would imply that any C^1 perturbation of ϕ would continue to have four balance points. That is, ϕ would be in the interior of $\text{BDiff}_+^1(\mathbb{S}^1, 4)$. \square

Corollary 4.8. *If $k \geq 1$ and $\phi \in \partial \text{BDiff}_+^{k,\alpha}(\mathbb{S}^1, 4)$, then ϕ has at least one pair of unstable balance points.*

Proof. If $\phi \in \partial \text{BDiff}_+^{k,\alpha}(\mathbb{S}^1, 4)$ for $k \geq 1$, then $\phi \in \partial \text{BDiff}_+^1(\mathbb{S}^1, 4)$, \square

We next introduce the appropriate energy space for \mathcal{E}_G^* – we work with this functional as it has nicer analytic properties. Define

$$H_{2\pi}^1(\mathbb{S}^1) = \left\{ u \in H^1(\mathbb{S}^1) : \int_{\mathbb{S}^1} e^u \, d\theta = 2\pi \right\} \subset H^1(\mathbb{S}^1).$$

The Sobolev embedding theorem implies $H^1(\mathbb{S}^1) \subset C^{1/2}(\mathbb{S}^1)$. Hence, $H_{2\pi}^1(\mathbb{S}^1)$ is a closed subset of $H^1(\mathbb{S}^1)$ with respect to the weak topology of $H^1(\mathbb{S}^1)$. Denote by

$$\text{HDiff}_+(\mathbb{S}^1) = \left\{ \phi \in \text{Diff}_+^1(\mathbb{S}^1) : \log \phi' \in H_{2\pi}^1(\mathbb{S}^1) \right\} \subset \text{Diff}_+^{1,1/2}(\mathbb{S}^1).$$

with strong (resp. weak) topology determined by $\phi_i \rightarrow \phi$ provided $\log \phi'_i \rightarrow \log \phi'$ strongly in $H^1(\mathbb{S}^1)$ (resp. weakly in $H^1(\mathbb{S}^1)$). For $\phi \in \text{HDiff}_+(\mathbb{S}^1)$ write

$$\mathcal{E}_G^*[\phi] = \int_{\mathbb{S}^1} \frac{1}{4} \left(\frac{\phi''}{\phi'} \right)^2 - (\phi')^2 \, d\theta + 2\pi$$

so for $\sigma \in \mathcal{D}_+^3$, $\mathcal{E}_G^*[\sigma] = \mathcal{E}_G^*[\phi_\sigma]$. We will need the following smoothing lemma:

Lemma 4.9. *For $\phi \in \text{HDiff}_+(\mathbb{S}^1)$, there exists a sequence $\phi_i \in \text{Diff}_+^\infty(\mathbb{S}^1)$ with $\phi_i \rightarrow \phi$ in the strong topology of $\text{HDiff}_+(\mathbb{S}^1)$. Furthermore, if ϕ satisfies $\phi \circ I = I \circ \phi$, then the ϕ_i may be chosen so $\phi_i \circ I = I \circ \phi_i$.*

Proof. Fix $p_0 \in \mathbb{S}^1$, let $\nu_\epsilon(p, p_0)$ be a family of C^∞ bump functions with $\nu_\epsilon(p, p_0) \geq 0$, and for fixed p_0 , $\lim_{\epsilon \rightarrow 0} \nu_\epsilon(p, p_0) = \delta_{p_0}$ and $\int_{\mathbb{S}^1} \nu_\epsilon(\cdot, p_0) \, d\theta = 1$. Set

$$P_\epsilon(p) = \int_{\mathbb{S}^1} \nu_\epsilon(p, \cdot) \phi'(\cdot) \, d\theta.$$

One checks that $P_\epsilon > 0$ and $\int_{\mathbb{S}^1} P_\epsilon \, d\theta = 2\pi$. It follows, that there are $\phi_\epsilon \in \text{Diff}_+^\infty(\mathbb{S}^1)$ so that $\phi_\epsilon(p_0) = \phi(p_0)$ and $\phi'_\epsilon = P_\epsilon$. Furthermore, if $\phi \circ I = I \circ \phi$, then $\phi' \circ I = \phi'$ and hence one computes that $P_\epsilon \circ I = P_\epsilon$. In particular, if $\phi \circ I = I \circ \phi$, then $\phi_\epsilon \circ I = I \circ \phi_\epsilon$. Finally, by standard Sobolev theory (cf. [5, Section 4.2]) the $\log P_\epsilon$ converge in $H^1(\mathbb{S}^1)$ to $\log \phi'$ – that is, ϕ_ϵ converge strongly to ϕ in $\text{HDiff}_+(\mathbb{S}^1)$. \square

Corollary 4.10. *If $\phi \in \text{HDiff}_+(\mathbb{S}^1)$ satisfies $\phi \circ I = I \circ \phi$, then $\mathcal{E}_G^*[\phi] \geq 0$ with equality if and only if $\phi \in \text{Möb}(\mathbb{S}^1)$.*

Proof. By Lemma 4.9, there are a sequence of $\phi_i \in \text{Diff}_+^\infty(\mathbb{S}^1)$, with $\phi_i \circ I = I \circ \phi_i$ and $\phi_i \rightarrow \phi$ strongly in $\text{HDiff}_+(\mathbb{S}^1)$. In particular, $\mathcal{E}_G^*[\phi_i] \rightarrow \mathcal{E}_G^*[\phi]$. Set $\psi_i = \phi_i^{-1}$ and note that $\psi_i \circ I = I \circ \psi_i$. Further, let $\sigma_i \in \mathcal{D}_i^+$ have induced diffeomorphism ψ_i . By (3.3), Proposition 3.2 and Corollary 4.2,

$$\mathcal{E}_G^*[\phi_i] = \mathcal{E}_G^*[\sigma_i^*] = \mathcal{E}_G[\sigma_i] \geq 0.$$

Proving the desired inequality. If one has equality, then the inequality implies that ϕ is critical with respect to variations preserving the symmetry. It follows that ϕ is smooth and so $\phi \in \text{Möb}(\mathbb{S}^1)$ by Corollary 4.2. \square

A symmetrization argument and Corollary 4.10 imply:

Proposition 4.11. *If $\phi \in \text{HDiff}_+(\mathbb{S}^1) \cap \partial \text{BDiff}_+^1(\mathbb{S}^1, 4)$, then*

$$\mathcal{E}_G^*[\phi] \geq 0$$

with equality if and only if $\phi \in \text{Möb}(\mathbb{S}^1)$.

Proof. Let $\phi \in \text{HDiff}_+^+(\mathbb{S}^1) \cap \partial \text{BDiff}_+^1(\mathbb{S}^1, 4)$. As $\phi \in \text{Diff}_+^{1, \frac{1}{2}}(\mathbb{S}^1)$, Corollary 4.8 implies that ϕ has at least one unstable balance point p_0 . Let γ_\pm be the two components of $\mathbb{S}^1 \setminus \{p_0, I(p_0)\}$. Up to relabelling, we may assume that

$$\mathcal{E}_G^*[\phi] \geq 2 \left(\int_{\gamma_-} \frac{1}{4} (u')^2 - e^{2u} \, d\theta + 2\pi \right)$$

where $u = \log \phi'$. Now define

$$\hat{u}(p) = \begin{cases} u(p) & p \in \bar{\gamma}_- \\ u(I(p)) & p \in \gamma_+ \end{cases}$$

Here, $\bar{\gamma}_-$ is the closure of γ_- in \mathbb{S}^1 . Clearly, \hat{u} is continuous, $\int_{\mathbb{S}^1} e^{\hat{u}} d\theta = 2\pi$ and

$$\mathcal{E}_G^*[\phi] \geq \int_{\mathbb{S}^1} \frac{1}{4} |\nabla \hat{u}|^2 - e^{2\hat{u}} d\theta + 2\pi.$$

Hence, there is a $\hat{\phi} \in \text{HDiff}_+(\mathbb{S}^1)$ so that $\hat{u} = \hat{\phi}'$. Notice that $\hat{\phi} \circ I = I \circ \hat{\phi}$ and so by Lemma 4.10

$$\mathcal{E}_G^*[\phi] \geq \mathcal{E}_G^*[\hat{\phi}] \geq 0,$$

with equality if and only if $\hat{\phi} \in \text{Möb}(\mathbb{S}^1)$. In the case of equality for ϕ we could reflect either γ_+ or γ_- , hence the preceding argument implies, $\phi|_{\gamma_{\pm}} = \psi_{\pm}$ for $\psi_{\pm} \in \text{Möb}(\mathbb{S}^1)$. However, a straightforward computation gives that if $\psi_+ \neq \psi_-$, then $\mathcal{E}_G^*[\phi] > 0$ – concluding the proof. \square

Finally, we will need to understand certain critical points of $\mathcal{E}_G^*[\phi]$.

Proposition 4.12. *Fix $\gamma \geq 2\pi$. If $u \in C^\infty(\mathbb{S}^1)$ satisfies the ODE*

$$(4.1) \quad \frac{1}{4} u'' - \alpha e^{2u} + \beta e^u = 0$$

and the constraints

$$(4.2) \quad \int_{\mathbb{S}^1} e^u d\theta = 2\pi \text{ and } \int_{\mathbb{S}^1} e^{2u} d\theta = \gamma,$$

then either $\gamma = 2\pi$, $\alpha = \beta$ and $u \equiv 0$ or $\gamma > 2\pi$ and there is an $n \in \mathbb{N}$ so that $\alpha = -n^2$ and $\beta = -\frac{\gamma}{2\pi} n^2$ and

$$u(p) = \frac{\gamma}{2\pi} + \sqrt{\left(\frac{\gamma}{2\pi}\right)^2 - 1} \cos(n(\theta(p) - \theta_0))$$

for some θ_0 . In this case,

$$\int_{\mathbb{S}^1} \frac{1}{4} (u')^2 - e^{2u} d\theta = -2\pi \frac{n^2}{4} + \frac{(n^2 - 4)}{4} \gamma.$$

Hence, if $n \geq 2$, then

$$\int_{\mathbb{S}^1} \frac{1}{4} (u')^2 - e^{2u} d\theta \geq -2\pi.$$

Proof. Set

$$E = E[u] = \int_{\mathbb{S}^1} \frac{1}{4} (u')^2 - e^{2u}.$$

It is a straightforward computation to see that (4.1) has the conservation law

$$\frac{1}{4} (u')^2 - \alpha e^{2u} + 2\beta e^u = \eta.$$

Integrating this we see that

$$2\pi\eta = E + (1 - \alpha)\gamma + 4\pi\beta.$$

However, integrating (4.1) gives that

$$-\alpha\gamma + 2\pi\beta = 0$$

and hence

$$E = 2\pi\eta - \gamma - 2\pi\beta.$$

Now set $U = e^{-u}$ one has that

$$\frac{1}{4} U'' = -\frac{1}{4} e^{-u} u'' + \frac{1}{4} e^{-u} (u')^2 = -\alpha e^u + \beta + \alpha e^u - 2\beta + \eta e^{-u} = \eta U - \beta$$

That is, U satisfies

$$U'' - 4\eta U = -4\beta.$$

As $U \in \mathbb{S}^1$ this immediately implies that either $U = \frac{\beta}{\eta}$, or $4\eta = -n^2$ for some $n \in \mathbb{Z}^+$ and

$$U = \frac{\beta}{\eta} + C_1 \cos \sqrt{-4\eta}\theta + C_2 \sin \sqrt{-4\eta}\theta$$

for some constants C_1, C_2 . Notice that in the first case, the constraints force

$$\eta = \beta$$

So $u = 0$ and hence $\alpha = \beta = \eta$, $\gamma = 2\pi$ and $E = -2\pi$.

In the second case, we note first that $U > 0$ and so

$$\frac{\beta}{\eta} > \sqrt{C_1^2 + C_2^2}.$$

Using the calculus of residues, we compute that

$$\begin{aligned} \int_{\mathbb{S}^1} e^u d\theta &= \int_{\mathbb{S}^1} \frac{1}{U} d\theta \\ &= \int_{\mathbb{S}^1} \frac{1}{\frac{\beta}{\eta} + \frac{C_1}{2}(z^n + z^{-n}) + \frac{C_2}{2i}(z^n - z^{-n})} \frac{dz}{iz} \\ &= \frac{2\pi}{\sqrt{\left(\frac{\beta}{\eta}\right)^2 - C_1^2 - C_2^2}} \end{aligned}$$

Hence, keeping in mind that $U > 0$, the first constraint is satisfied if and only if

$$\beta = \eta \sqrt{1 + C_1^2 + C_2^2}.$$

Hence,

$$u = -\log \left(\sqrt{1 + C_1^2 + C_2^2} + C_1 \cos \sqrt{-4\eta}\theta + C_2 \sin \sqrt{-4\eta}\theta \right).$$

Plugging this into (4.1), shows that $\alpha = \eta$. Hence,

$$\gamma = 2\pi\beta/\alpha = 2\pi\sqrt{1 + C_1^2 + C_2^2}.$$

We conclude that,

$$E = 2\pi\eta - \gamma - \eta\gamma = -2\pi\frac{n^2}{4} + \left(\frac{1}{4}n^2 - 1\right)\gamma.$$

Hence, if $n \geq 2$, then as $\gamma \geq 2\pi$

$$E \geq -2\pi\frac{n^2}{4} + 2\pi\left(\frac{n^2}{4} - 1\right) \geq -2\pi.$$

with equality if and only if $n = 2$. □

Remark 4.13. If $n = 1$, then by taking γ large we can make E as small as we like.

Combining Propositions 4.11 and 4.12 gives:

Proposition 4.14. *If $\phi \in \text{HDiff}_+(\mathbb{S}^1) \cap \overline{\text{BDiff}}_+^1(\mathbb{S}^1, 4)$, then*

$$\mathcal{E}_G^*[\phi] \geq 0$$

with equality if and only if $\phi \in \text{Möb}(\mathbb{S}^1)$.

Remark 4.15. This result is sharp in that the inequality fails for the explicit elements in $\text{BDiff}_+^\infty(\mathbb{S}^1, 2)$ above.

Proof. Suppose that this was not the case, that is there was some $\phi_0 \in \text{HDiff}_+(\mathbb{S}^1) \cap \overline{\text{BDiff}}_+^1(\mathbb{S}^1, 4)$ so that $\mathcal{E}_G^*[\phi_0] < 0$. Let $u_0 = \log \phi'_0$ and set $\gamma_0 = \int_{\mathbb{S}^1} (\phi')^2 d\theta = \int_{\mathbb{S}^1} e^{2u_0} d\theta$. The Cauchy-Schwarz inequality implies that $\gamma_0 \geq 2\pi$ with equality if and only if $u_0 \equiv 0$. Now consider the minimization problem

$$(4.3) \quad E(\gamma) = \inf \left\{ \mathcal{E}_G^*[\phi] \mid \phi \in \text{HDiff}_+(\mathbb{S}^1) \cap \overline{\text{BDiff}}_+^1(\mathbb{S}^1, 4), \int_{\mathbb{S}^1} (\phi')^2 d\theta = \gamma \right\}.$$

Clearly, our assumption ensures that $E(\gamma_0) \leq \mathcal{E}_G^*[\phi_0] < -2\pi$. Rellich compactness, implies that, there is a u_{\min} so that

$$E(\gamma_0) = \int_{\mathbb{S}^1} \frac{1}{4} (u'_{\min})^2 - e^{2u_{\min}} d\theta + 2\pi = \int_{\mathbb{S}^1} \frac{1}{4} (u'_{\min})^2 d\theta - \gamma_0 + 2\pi < 0.$$

and a $\phi_{\min} \in \text{HDiff}_+(\mathbb{S}^1) \cap \overline{\text{BDiff}}_+^1(\mathbb{S}^1, 4)$ so that $\log \phi'_{\min} = u_{\min}$. However, Proposition 4.11 implies that $\phi_{\min} \in \text{BDiff}_+^1(\mathbb{S}^1, 4)$. This implies that u_{\min} is critical with respect to arbitrary variations in $H^1(\mathbb{S}^1)$ which preserve the constraints

$$\int_{\mathbb{S}^1} e^u d\theta = 2\pi \quad \text{and} \quad \int_{\mathbb{S}^1} e^{2u} d\theta = \gamma_0.$$

Hence, u_{\min} is smooth and satisfies the Euler-Lagrange equation

$$\frac{1}{4} u''_{\min} + C_1 e^{2u_{\min}} + C_2 e^{u_{\min}} = 0.$$

Applying Proposition 4.12 to u_{\min} , contradicts $E(\gamma_0) < 0$ proving the result. \square

We may now conclude the main geometric estimates.

Proof of Theorem 1.1. The natural scaling of the problem means that we may apply a homothety to take $L(\sigma) = 2\pi$. As σ is a smooth closed strictly convex curve, it is a smooth degree-one strictly convex curve. Let $\phi_\sigma \in \text{Diff}^+(\mathbb{S}^1)$, be the induced diffeomorphism and let $\psi_\sigma = \phi_\sigma^{-1}$. By Corollary 4.6, $\phi_\sigma, \psi_\sigma \in \text{BDiff}_+^1(\mathbb{S}^1, 4)$. The claim now follows from Propositions 4.14 and 3.2. \square

APPENDIX A. AN EXTENDED OVALS CONJECTURE

As a final remark, we note that as originally posed, the ovals problem concerned closed curves. In light of the present paper, it seems reasonable to extend the conjecture to smooth degree-one curves with more than two balance points.

Conjecture A.1. *If $\sigma \in \mathcal{D}_+^\infty$ has the property that the induced diffeomorphism $\phi_\sigma \in \text{BDiff}_+^\infty(\mathbb{S}^1, 4)$, then for all $f \in C^\infty(\mathbb{S}^1)$,*

$$\mathcal{E}_S[\sigma, f] \geq 0.$$

As $\phi \in \text{BDiff}_+^\infty(\mathbb{S}^1, 4)$ if and only if $\phi^{-1} \in \text{BDiff}_+^\infty(\mathbb{S}^1, 4)$, Proposition 3.4 implies this conjecture is equivalent to showing that for such curves

$$\mathcal{E}_S^*[\sigma, f] \geq 0.$$

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